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A NUMERICAL METHOD FOR ELASTIC–PLASTIC TORSION BY VARIATIONAL INEQUALITY

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Abstract—The paper is concerned with a numerical procedure to solve the elastic–plastic torsion of a bar using the variational inequality concept. The finite element method with SOR (successive over relaxation) technique is employed to solve the variational inequality as a constrained minimuzation problem. Circular, square, L-shaped cross sections, and square cross section with a slit are considered as numerical examples. Von Mises yield criteria is employed.

1. INTRODUCTION

The problem of elastic–plastic torsion of a bar twisted by terminal couples can be formulated in terms of a stress function ϕ. The elastic–plastic torsion problem involves finding ϕ such that

$$\nabla^2\phi = -f \qquad \text{in } \Omega_e \tag{1}$$

$$|\nabla\phi|^2 - \tau_p{}^2 = 0 \qquad \text{in } \Omega_p \tag{2}$$

$$|\nabla\phi|^2 - \tau_p{}^2 < 0 \qquad \text{in } \Omega_e \tag{3}$$

$$\phi = 0 \qquad \text{on } \partial\Omega \tag{4}$$

$$\frac{\partial\phi}{\partial n_e} = \frac{\partial\phi}{\partial n_p} \qquad \text{on } \Gamma \tag{5}$$

$$T = \int_\Omega 2\phi \, \mathrm{d}x \, \mathrm{d}y \tag{6}$$

where Ω_e and Ω_p denote the elastic and plastic portions of the domain (i.e. cross section of the bar) $\Omega = \Omega_e U \Omega_p$, $\partial\Omega$ denotes piecewise smooth boundary, Γ is the interface of elastic and plastic portion, $(\partial/\partial n_e)$ and $(\partial/\partial n_p)$ are outer normal derivative on Γ with respect to Ω_e and Ω_p, τ_p is a given plastic (shear) stress, and $f = 2G\theta$; here G is the shear modulus, θ is the angular twist per unit length and T is the applied torque on the bar. Note that eqn (3) is an inequality constraint on the gradient of ϕ.

The first complete description of the elastic–plastic torsion is apparently due to von Mises[1]. Existence and uniqueness of solution to elastic–plastic torsion of a square cross section bar was established by Ting[2] using a direct approach on eqns (1)–(6). A different approach, based on the theory of variational inequalities, was taken by Stampacchia[3], Lions[4] and Duvaut and Lions[5]. In the theory of variational inequalities the problem in (1)–(6) is formulated as a constrained (which is an inequality constraint) minimization problem on a restricted function space. In [3–5] existence, uniqueness and regularity of solutions to the elastic–plastic torsion was considered.

A vast majority of papers on variational inequalities dwell on theoretical aspects such as existence and uniqueness of solutions, and very little can be found on computational and numerical results. Numerical solutions to the elastic–plastic torsion problem have been obtained using finite difference methods, relaxation methods, nonlinear programming methods and finite element methods. Much of the previous works have used direct formulations based on (1)–(6). While the theory of variational inequalities is a natural (and perhaps the most correct) means of formulating the elastic–plastic torsion problem, its numerical implementation is by no means easy. This fact is reflected by the very few publications on the application of variational inequalities to engineering problems. Numerical solution of the elastic–plastic torsion by the variational inequality is a recent occurrence. Glowinski and his colleagues[6–9] presented numerical results for various twist angles for simply and multiconnected domains using the Uzawa's method. Recently, Tabata[10] presented a numerical method for the variational inequality associated with the elastic–plastic torsion problem. There the finite element method was used to reduce the problem to a minimization problem, which was then solved by the interior point unconstrained minimization technique[11] with the steepest descent method and the generalized Newton method. In [11] it is pointed out that the gradient method, which is the basis of Uzawa's method, has slow convergence compared to the interior point unconstrained minimization technique

The present investigation is concerned with a new numerical procedure to solve the variational inequality associated with the elastic–plastic torsion problem (1)–(6). The main difficulty with the problem is to locate the elastic–plastic interface. In [6–8], eqns (2) and (3) are treated as constraints by means of a Lagrange multiplier, and it is assumed that the angle of twist is given instead of the terminal torque. In the present investigation, eqn (2) is solved by means of the method of characteristics, and then an iterative procedure is used to solve for the elastic–plastic interface. We also reformulate the problem in [6–8] for the case in which the torque is given. The present investigation is primarily concerned with a new and effective computational scheme to solve for the elastic–plastic torsion; nevertheless, theoretical aspects, such as the existence and uniqueness of solutions and error estimates are also discussed for the sake of completeness. A number of numerical examples are presented assuming that von Mises' yield criteria holds.

2. VARIATIONAL FORMULATIONS

Associated with the elastic–plastic torsion problem (1)–(6), we discuss two methods; one due to [6–8], called Problem L, and the other is the new method which we will call Problem P. In each method we consider two cases: In case 1, the angle of twist is assumed to be given and in case 2 the torque is assumed to be known. We use the following notation. Let $L_2(\Omega)$ denote the space of square integrable functions in Ω and $H_0^1(\Omega)$ denote the Sobolev space of order 1 with compact support in Ω (i.e. $H_0^1(\Omega)$ contains functions which, along with their first derivatives, belong to $L^2(\Omega)$, and vanish on the boundary of Ω).

Problem 1

When f $(=2G\theta)$ is given, (1)–(5) are formulated as a minimization problem on the restricted function space K:

($P1$). Find $\phi \in K$ such that for given $f \in L^2(\Omega)$

$$F(\phi) \leq F(\bar{\phi}) \quad \text{for} \quad \forall \bar{\phi} \in K \tag{7}$$

where $F(\phi) = \int_\Omega (|\nabla\phi|^2 - 2f\phi)\,dx\,dy$, and K is the closed convex set.

$$K = \{\phi \in H_0^1 \colon |\nabla\phi|^2 - \tau_p{}^2 \leq 0 \text{ in } \Omega\}. \tag{8}$$

Here torque T can be computed from (6) once ϕ is known. It is easy to see that the problem P1 is a minimization problem on set K and is equivalent to the problem described by (1)–(5). To show this, first note that eqns (2)–(4) are included in the definition of K, and it remains to be shown that P1 is equivalent to (1) and (5). Suppose that $\phi_0 \in K$ minimizes $F(\phi)$ on K. Then for $\phi_0, \bar{\phi} \in K$,

$$\delta F(\phi_0, \bar{\phi} - \phi_0) \geq 0$$

or, equivalently, with $\eta = \bar{\phi} - \phi_0$,

$$\text{line}_{\epsilon\to 0} \frac{1}{\epsilon}(F(\phi_0 + \epsilon\eta) - F(\eta)) = \frac{\partial}{\partial\epsilon} F(\phi_0 + \epsilon\eta)|_{\epsilon=0} \geq 0.$$

That is,

$$\frac{\partial}{\partial \epsilon}\int_{\Omega}\{|\nabla(\phi_0+\epsilon\eta)|^2+2f(\phi_0+\epsilon\eta)\}\,dx\,dy|_{\epsilon=0}\geq 0$$

$$\int_{\Omega}\{2\nabla\phi_0\cdot\nabla\eta-2f\eta\}\,dx\,dy\geq 0$$

$$-\int_{\Omega_e}(\nabla^2\phi_0+f)\eta\,dx\,dy-\int_{\Omega_p}(\nabla^2\phi_0+f)\eta\,dx\,dy+\int_{\partial\Omega_e}\frac{\partial\phi_0}{\partial n}\eta\,ds+\int_{\partial\Omega_p}\frac{\partial\phi_0}{\partial n}\eta\,ds\geq 0$$

$$-\int_{\Omega_e}(\nabla^2\phi_0+f)\eta\,dx\,dy-\int_{\Omega_p}(\nabla^2\phi_0+f)\eta\,dx\,dy+\int_{\partial\Omega_e/\Gamma}\frac{\partial\phi_0}{\partial n}\eta\,ds$$

$$+\int_{\partial\Omega_p/\Gamma}\frac{\partial\phi_0}{\partial n}\eta\,ds+\int_{\Gamma}\left\{\left(\frac{\partial\phi_0}{\partial n}\eta\right)_e-\left(\frac{\partial\phi_0}{\partial n}\eta\right)_p\right\}ds\geq 0$$

where $\Gamma=\Omega_e\cap\Omega_p$ is the interface of the elastic and plastic regions. Since $\eta=\bar{\phi}-\phi_0=0$ in Ω_p, and η can be positive or negative in Ω_e and $\eta=0$ on $\partial\Omega=\partial\Omega_e\cup\partial\Omega_p$, we have

$$\nabla^2\phi_0+f=0\quad\text{in }\Omega_e,$$
$$\left(\frac{\partial\phi_0}{\partial n}\right)_e-\left(\frac{\partial\phi_0}{\partial n}\right)_p=0\quad\text{on }\Gamma$$

which are the same as (1) and (5).

It is useful to cast Problem 1 in alternate but equivalent forms:

($P1.1$). Find $\phi\in K$ such that for $f\in L^2(\Omega)$,

$$B(\phi,\bar{\phi}-\phi)\geq(f,\bar{\phi}-\phi),\quad\text{for }\forall\bar{\phi}\in K \tag{9}$$

where

$$B(\phi,\bar{\phi})=\int_{\Omega}\nabla\phi\cdot\nabla\bar{\phi}\,dx\,dy,\ (f,\bar{\phi})=\int_{\Omega}f\bar{\phi}\,dx\,dy. \tag{10}$$

Equation (9) is the variational inequality associated with Problem 1.

Problem 2

Here we assume that the torque T is given, but f is unknown. Introducing a Lagrange multiplier λ, we can state (1)–(6) as a saddle point problem:

($P2$). Find $(\phi,\lambda)\in K\times R$ such that, for given $T\in R$,

$$G(\phi,\bar{\lambda})\leq G(\phi,\lambda)\leq G(\bar{\phi},\lambda) \tag{11}$$

for every $(\bar{\phi},\bar{\lambda})\in K\times R$. Here $G(\phi,\lambda)$ is given by

$$G(\phi,\lambda)=\int_{\Omega}|\nabla\phi|^2\,dx\,dy+\lambda\left(T-\int_{\Omega}2\phi\,dx\,dy\right) \tag{12}$$

and R denotes the set of real numbers. It can be verified that P2 is equivalent to (1)–(6). The left inequality in (11) gives eqn (6) and the right inequality gives (1)–(5). Alternate formulations of (11) are possible, as stated below:

($P2.1$). Find $(\phi,\lambda)\in K\times R$ such that, for given $T\in R$,

$$B(\phi,\bar{\phi}-\phi)\geq(\lambda,\bar{\phi}-\phi)\quad\text{and}\quad T=\int_{\Omega}2\phi\,dx\,dy \tag{13}$$

for every $\bar{\phi}\in K$.

Note that Problems 1 and 2 are formulated on a special set K. From numerical point of view, it is difficult to construct a finitedimensional subspace of K. However, by solving eqn (2) (say, by the method of characteristics), the set K can be identified. A general formulation would be to include eqns (2) and (3) as a constraint by means of a Lagrange multiplier (see [6–8]).

Problem 3

Problem 3 seeks to find a solution to (1)–(6) over a larger set (instead of K).

($L1$). Find $(\phi, P) \in H_0^1(\Omega)\times N$ such that, for given $f \in L^2(\Omega)$,

$$L(\phi, \bar{P}) \le L(\phi, P) \le L(\bar{\phi}, P) \tag{14}$$

for every $(\bar{\phi}, \bar{P}) \in H_0^1(\Omega)\times N$, where

$$L(\phi, P) = F(\phi) - \int_\Omega P(|\nabla\phi|^2 - \tau_p^2)\,\mathrm{d}x\,\mathrm{d}y \tag{15}$$

$$N = \{q \in L^\infty(\Omega):\ q \le 0 \text{ in } \Omega\}.$$

It is not difficult to show that Problem 3 is equivalent to Problem 1, and hence to eqns (1)–(5). Once again, alternate formulations to (14) are possible.

($L1.1$). Find $(\phi, P) \in H_0^1(\Omega)\times N$ such that for given $f \in L^2(\Omega)$

$$B_p(\phi, \bar{\phi}) = (f, \bar{\phi}), \quad \forall\bar{\phi} \in H_0^1(\Omega) \tag{16}$$

$$\int_\Omega \{|\nabla\phi|^2 - \tau_p^2\}(\bar{P} - P)\,\mathrm{d}x\,\mathrm{d}y \ge 0, \ \forall\bar{P} \in N \tag{17}$$

where

$$B_p(\phi, \bar{\phi}) = \int_\Omega (1 - P)\nabla\phi\cdot\nabla\bar{\phi}\,\mathrm{d}x\,\mathrm{d}y. \tag{18}$$

Problem 4

Problem 4 is the same as Problem 3, except that instead of given f, T is given.

($L2$). Find $(\phi, \lambda, P) \in H_0^1(\Omega)\times R\times N \equiv \chi$ such that

$$Q(\phi, \lambda, \bar{P}) \le Q(\phi, \lambda, P) \le Q(\bar{\phi}, \lambda, P) \tag{19}$$

$$Q(\phi, \bar{\lambda}, P) \le Q(\phi, \lambda, P) \le Q(\bar{\phi}, \lambda, P) \tag{20}$$

for $T \in R$, and every $(\bar{\phi}, \bar{\lambda}, \bar{P}) \in \chi$, where

$$Q(\phi, \lambda, P) = \int_\Omega |\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y + \lambda\left(T - \int_\Omega 2\phi\,\mathrm{d}x\,\mathrm{d}y\right) - \int_\Omega P(|\nabla\phi|^2 - \tau_p^2)\,\mathrm{d}x\,\mathrm{d}y. \tag{21}$$

Equivalence of Problem 4 to the other formulations given herein can be established. Problem 4 is the most general (and less restrictive) one for the elastic–plastic torsion problem. Equivalent alternate forms of (19) and (20) are given by

($L2.1$). Find $(\phi, \lambda, p) \in \chi$ such that

$$B_p(\phi, \bar{\phi}) = (\lambda, \bar{\phi}), \quad T - \int_\Omega 2\phi\,\mathrm{d}x\,\mathrm{d}y = 0, \quad T \in R \tag{22}$$

$$\int_\Omega (|\nabla\phi|^2 - \tau_p^2)(\bar{P} - P)\,\mathrm{d}x\,\mathrm{d}y \ge 0 \tag{23}$$

for every $(\bar{\phi}, \bar{\lambda}, \bar{p}) \in \chi$.

Existence and uniqueness

Existence and uniqueness of solutions to variational inequalities have been established by Lions and Stampacchia[12] for coersive and non-negative forms. Since the variational inequalities considered here are special cases of those in [12], existence and uniqueness of solutions follow immediately. Here we state a theorem on the existence and uniqueness of solutions to a general variational inequality. Problem 1 is a special case of it.

Theorem 1 (*existence and uniqueness*). Let $B(u, v)$ be a bilinear form on H (not necessarily symmetric) satisfying the conditions,

(i) $$|B(u, v)| \leq C\|u\|\,\|v\| \text{ for } u, v \in H \text{ (continuity)} \tag{24}$$

(ii) $$|B(u, u)| \geq \mu\|u\|^2 \quad \mu > 0 \text{ (coersivity)} \tag{25}$$

where H is a Hilbert space with inner product $((.,.))$ and norm $\|.\|$. Let K be a closed convex set of H, and f be an element of the dual H' of H. Then there exists a unique solution to the problem of seeking $u \in K$ such that

$$B(u, v-u) \geq (f, v-u) \quad \text{for} \quad v \in K \tag{26}$$

where $(.,.)$ denotes the duality pairing between H' and H.

Proof. Proof of this theorem is given in [12]. Here we prove only the uniqueness. Let u_1 and u_2 be the solutions of (26) in K. Then

$$B(u_1, v-u_1) \geq (f, v-u_1)$$
$$B(u_2, v-u_2) \geq (f, v-u_2).$$

By setting $v = u_2$ in the first and $v = u_1$ in the second and adding we have

$$-B(u_1-u_2, u_1-u_2) \geq 0 \quad \text{or} \quad B(u_1-u_2, u_1-u_2) \leq 0.$$

In view of the coersivity (25), it follows that $\|u_1 - u_2\| = 0$ or $u_1 = u_2$.

3. NUMERICAL PROCEDURE AND EXAMPLES

Solution procedure. Here we describe the solution procedure for Problems P1 and P2. Solution procedures for Problems L1 and L2 can be found in [6–8]. In Problems 1 and 2, the convex set K is unknown for numerical approximation point of view. An alternate way to find the elements of K is to seek elements from an equivalent set, $\bar{K}$

$$\bar{K} = \{\phi \in H_0^1(\Omega)\colon\ |\phi| \leq |\phi_p|\} \tag{27}$$

wherein ϕ_p is determined by solving the equation

$$|\nabla\phi_p|^2 - \tau_p^2 = 0. \tag{28}$$

The modified problem enables the numerical solution of P1 and P2, provided (28) can be solved for ϕ_p.

Note that ϕ_p is the solution of the fully-plastic case. Prager and Hodge[13] and Nadai[14] have given, in connection with the fully-plastic torsion problem, a physical construction of ϕ_p as a surface of constant slope belonging to a given cross section. Nadai[14] demonstrated, by means of sandhills, the solution to the fully-plastic problem (28). It is recognized that (28) is a hyperbolic equation (see, e.g. Geiringer[15]), and its solution can be obtained by means of the theory of characteristics. Here, we exploit this idea to construct the numerical scheme to solve Problems 1 and 2. Details of the methodology are given in the Appendix. The solution procedure can be summarized as follows:

(i) Solve eqn (28) using the theory presented in the Appendix.

(iia) When f is given (like in Problem P1) solve eqn (10). The finite dimensional form (or numerical analogue) of (10) is obtained by the use of the finite element method:

$$\{\phi\} = P_K{}^h[\{\phi\} - \rho([K]\{\phi\} - \{F\})], \qquad \rho > 0 \tag{29}$$

where K_{ij} is the stiffness matrix, and F_i is the load vector,

$$K_{ij} = B(N_i, N_j), \quad F_i = (f, N_i).$$

Here N_i denotes the shape function, and $P_K{}^h$ is the projection operator from finite-dimensional subspace $S_h \subset H_0{}^1(\Omega)$ to $S_h \cap K \equiv K_h$. This is equivalent to finding the minimum of the set (the quantity in the square brackets, the solution of (28)). Note that eqn (29) must be solved iteratively: for $f > 0$

$$\{\phi\}_{n+1} = \min\{[\{\phi\}_n - \rho([K]\{\phi\}_n - \{F\})], \{\phi_p\}\}. \tag{30}$$

At the beginning of the iterative procedure, one can choose $\{\phi\}_0$ as a zero vector.

(iib) When T is given (like in problem P2), the same procedure outlined in (iia) must be followed for $(13)_1$:

$$\{\phi\}_{n+1} = \min\{[\{\phi\}_n - \rho_\phi([K]\{\phi\}_n - \{\Lambda\}_m)], \{\phi_p\}\}, \qquad \rho_\phi > 0 \tag{31}$$

where $\Lambda_i^m = (\lambda_m, N_i)$, λ_m being the Guess value at the mth iteration on the second equation in (13):

$$\lambda_{m+1} = \lambda_m - \rho_\lambda(T - \{\phi_\lambda\}^T\{g\}), \qquad \rho_2 > 0 \tag{32}$$

where $\{\phi_\lambda\}$ is the converged solution of (31) for λ_m and $\{g\}$ is the vector

$$g_i = \int_\Omega 2N_i \, dx \, dy.$$

At the beginning of the iteration procedure, a value for λ is assumed and eqn (31) is solved iteratively for $\{\phi_{\lambda_m}\}$ until it converges. Using $\{\phi_{\lambda_m}\}$, eqn (32) is solved for new λ_{m+1}. Using λ_{m+1} in (31) the procedure is repeated until the difference between two consecutive iteration values of λ differ by a small preassigned value.

Note that eqn (28) is solved only once for a given problem. The convergence of the iterative procedure in step (ii) depends on the relaxation parameters ρ, ρ_ϕ, ρ_λ, etc. More specific comments will be made later. The approximation error for ϕ, for fixed λ, is given by the usual error estimates (see Oden and Reddy[16]):

$$\|\phi_0 - \phi\|_{H^1(\Omega)} \leq Ch^{k-m+1}\|\phi_0\|_{H^k(\Omega)}$$

where ϕ_0 is the exact solution, and $2m$ is the order of the differential equation.

Numerical examples. First the circular cross section problem, for which the exact solution is available, is solved using all four formulations P1.1, P2.1, L1.1 and L2.1. The finite element mesh is shown in Fig. 1(a). A modified form of successive over relaxation (SOR) method is used to obtain the solution, and $\tau_p = 1$ is used in all examples.

The results are shown in Tables 1–3. Formulations P1.1 and P2.1 (present) give more accurate results for ϕ compared to those obtained by formulations L1.1 and L2.1. However, the square of the gradient of ϕ is more accurately computed by Problems L1.1 and L2.1. This is expected since in the latter case the gradient condition is included as part of the problem and therefore is satisfied more closely. However, Problems L1.1, and L2.1 took more computational time (almost twice) compared to Problems P1.1 and P2.1. Table 4 shows the solution for the refined mesh shown in Fig. 1(b). The elastic–plastic torsion solution for $f = 4$ using mesh (a) gives a torque of $T = 0.49146$. The elastic torsion problem is solved using $f = 4.0$ and also using

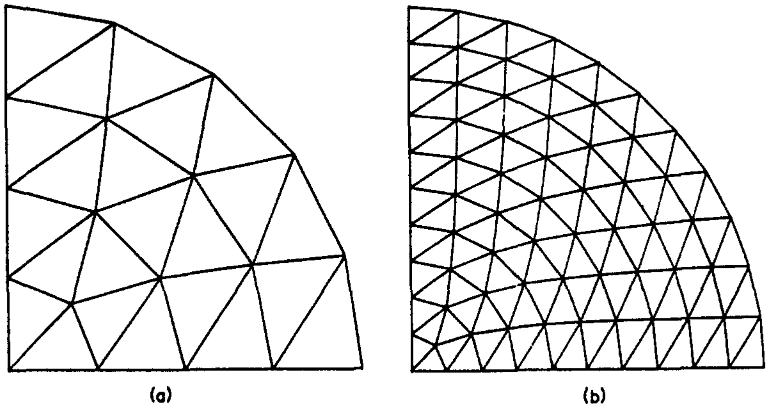


Fig. 1. Mesh for circular cross section with $r = 1.0$.

Table 1. Comparison of ϕ by various formulations for circular cross section

r \ Method	Exact	Method $P1.1$	Method $L1.1$	Method $P2.1$	Method $L2.1$
0.0	0.75	0.7564	0.7374	0.7564	0.7362
0.25	0.6875	0.6884	0.6703	0.6884	0.6695
0.5	0.5	0.5	0.4909	0.5	0.4911
0.75	0.25	0.25	0.2469	0.25	0.2470
1.0	0.0	0.0	0.0	0.0	0.0
Comparison of torque and the Lagrange multiplier†					
T	0.5072	0.4915	0.4748	(0.4915)	(0.4748)
λ	4.0	(4.0)	(4.0)	4.0003	3.9784

†Values in parenthesis indicate specified values.

Table 2. Comparison of $|\nabla\phi^2|$ by various formulations for circular cross section

r	Exact	Method $P1.1$	Method $L1.1$	Method $P2.1$	Method $L2.1$
0.125	0.0625	0.3099	0.3112	0.3097	0.3094
0.375	0.5625	0.7801	0.7610	0.7802	0.7568
0.625	1.0	1.09195	0.9999	1.0195	1.0005
0.875	1.0	1.0125	1.0001	1.0125	1.0003

Table 3. Comparison of percentage errors in various formulations for circular cross section bar

Quantity	Method $P1.1$	Method $L1.1$	Method $P2.1$	Method $L2.1$
ϕ_{max}	0.0086	0.0168	0.0086	0.0184
λ	—	—	0.0	0.0054
T	0.031	0.0639	0.031	0.0639

Table 4. Comparison of solution ϕ for circular cross section with fine mesh ($f = 4$)

r	Method $P1.1$ ϕ	τ	Exact solution ϕ	τ
0.0	0.7498	0.1082	0.75	0.1
0.1	0.7390	0.2962	0.74	0.3
0.2	0.7094	0.4960	0.71	0.5
0.3	0.6598	0.6975	0.66	0.7
0.4	0.5900	0.9002	0.59	0.9
0.5	0.5	1.0	0.5	1.0
0.6	0.4	1.0	0.4	1.0
0.7	0.3	1.0	0.3	1.0
0.8	0.2	1.0	0.2	1.0
0.9	0.1	1.0	0.1	1.0
1.0	0.0	1.0	0.0	1.0

$T = 0.49146$, and the solutions are plotted along with the elastic–plastic solution in Fig. 2. Note that the elastic solution given by the applied torque $T = 0.49146$ is redistributed due to the constant slope of ϕ in the plastic region.

The choice of the acceleration parameter ρ is very important for numerical convergence of the solution. It is shown by Young[17] that the acceleration parameter in SOR method for positive definite linear equations should be such that $0 < \rho < 2$ for convergence. To find the optimum value of ρ_ϕ in (30) on the convergence, we solved the elastic–plastic torsion of circular shaft for various values of ρ_ϕ. Figure 3 shows ϕ_{max} vs number of iterations for various values of ρ_ϕ between 0 and 2. However, $\rho_\phi = 1$ is the best value for the problem and ρ_ϕ tends to increase as the mesh parameter h decreases.

In the case of ρ_λ, Young's[17] estimate is not valid. In our tests we found that ρ_λ around 4.0 gives the fast rate of convergence. Figure 4(a) shows the log of the difference between values of λ in two consecutive iterations against the number of iterations. Similarly, it is found that ρ_p around 0.5 gives the faster convergence for P, the Lagrange multiplier. Figure 4(b) shows the log of the difference between values of P in two consecutive iterations against the number of iterations. Finally, Fig. 4(c) shows the total number of iterations required for convergence at

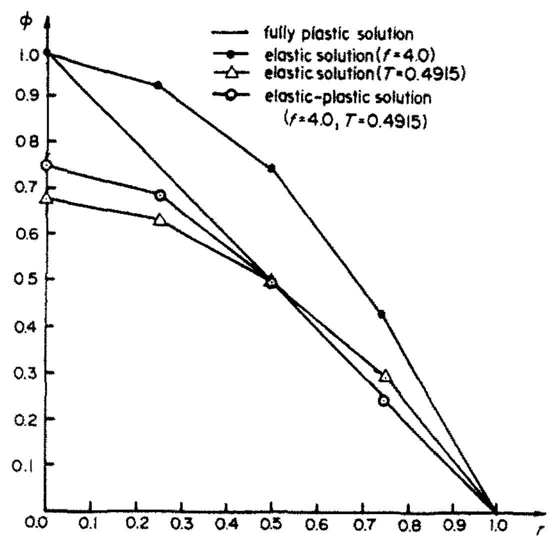


Fig. 2. Comparison of elastic and elastic–plastic solution for circular cross section.

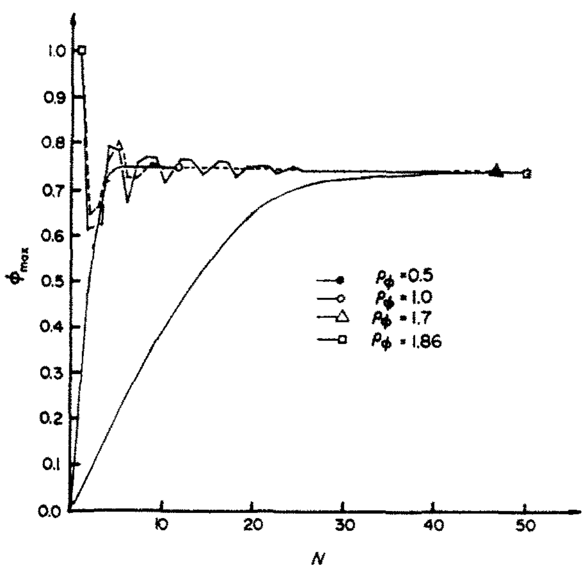


Fig. 3. Influence of the acceleration parameter on the convergence of ϕ_{max}.

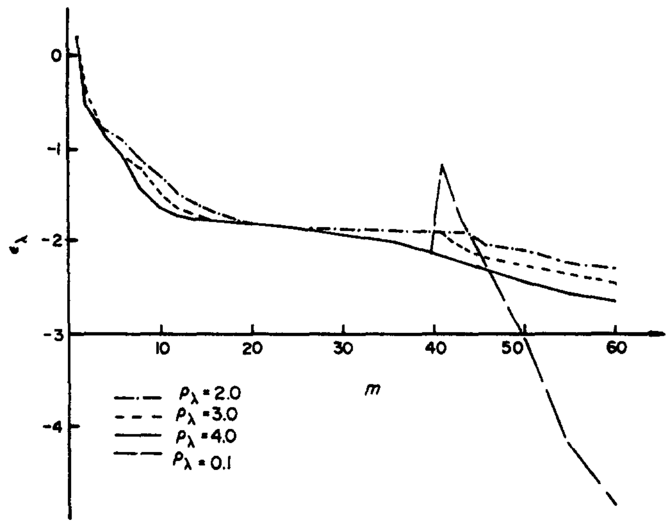


Fig. 4(a). Convergence of λ.

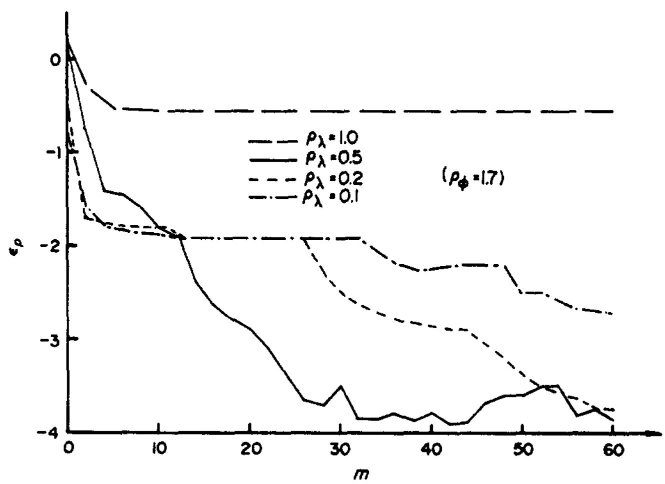


Fig. 4(b). Convergence of p.

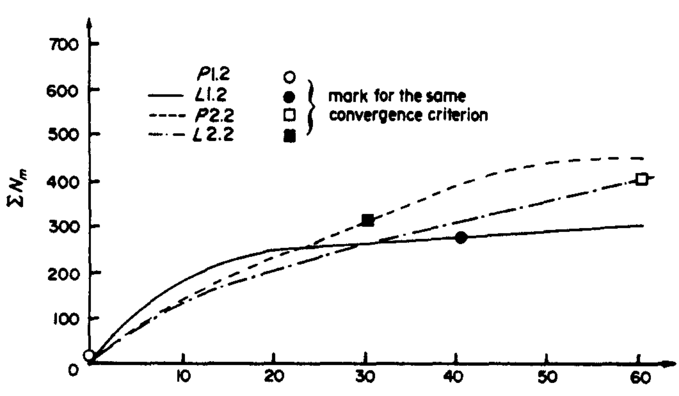


Fig. 4(c). The number of total iterations at the mth step.

mth iteration on λ and P for all four formulations (for circular cross section) using $\rho_\phi = 1.7$, $\rho_\lambda = 4.0$ and $\rho_p = 0.5$.

Using Formulation P1.1 several other examples are solved. These include square cross section L-shaped cross section, and square cross section with a slit. We have used $f = 2G\theta = 2, 4$ and 6 to compute the elastic–plastic solution. Biaxial symmetry is the square cross section case, and axial symmetry in the square cross section with a slit case are used in the finite element analysis.

Figure 5 shows the finite element mesh, and equi-stress function lines for $f = 2$, 4 and 6 for the square cross section problem. Figure 6 shows a composite picture of the plastic regions for

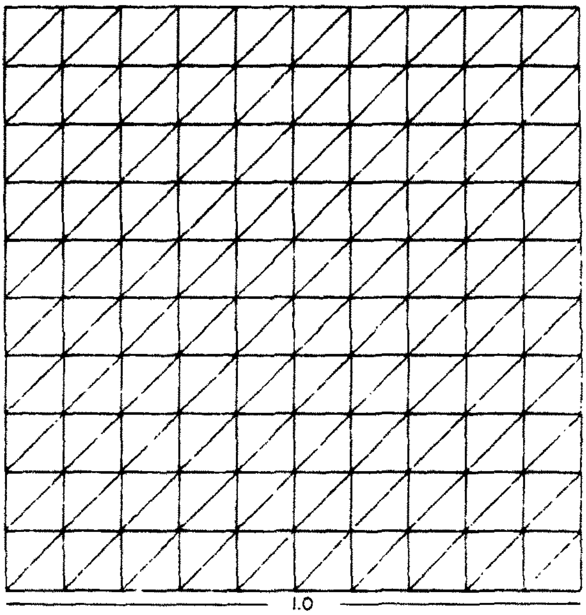


Fig. 5(a).

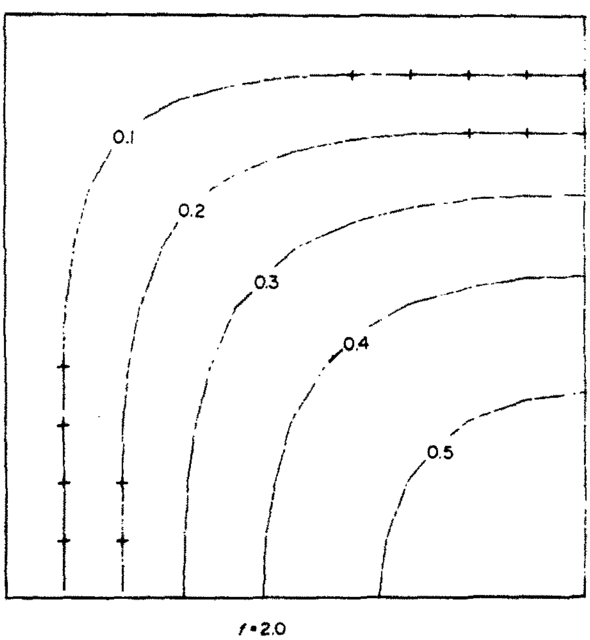
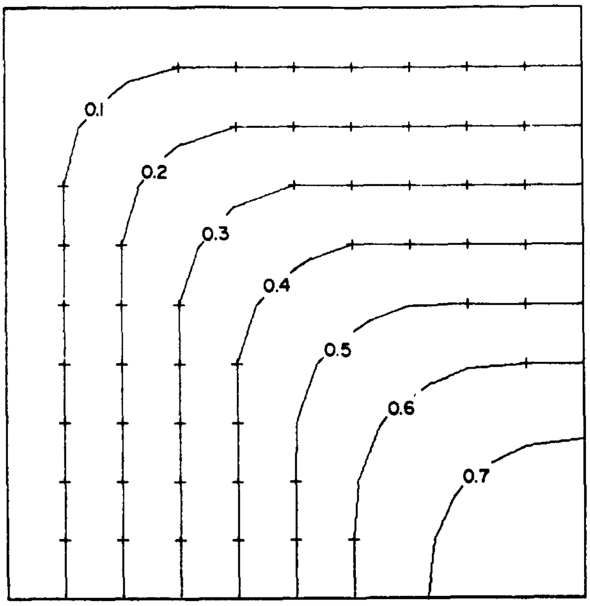
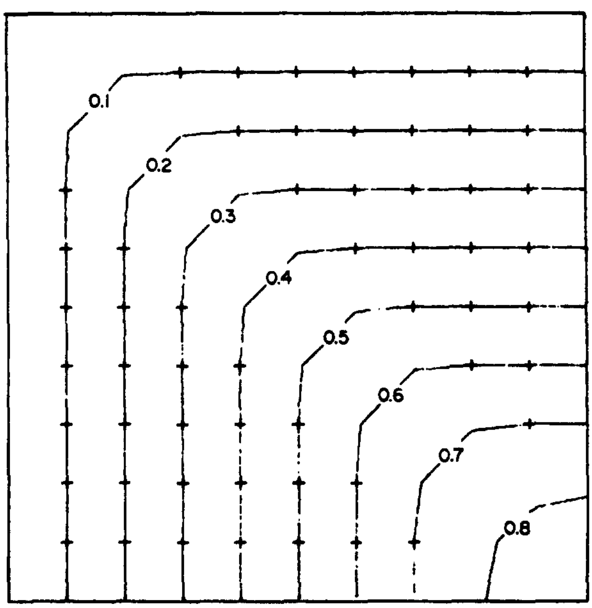


Fig. 5(b).



$f = 4.0$

Fig. 5(c).



$f = 6.0$

Fig. 5(d).

Figure 5. Equi-stress function lines and yielded portion for square cross section. + shows yielded nodes.

various values of f. Figure 7(a) shows the nonuniform finite element mesh for L-shaped cross section. Figures 7(b)–(d) show equi-stress function lines for $f = 2$, 4 and 6. The shear stress distribution is shown in Fig. 8. A composite picture of plastic regions is shown in Fig. 9. Finally, Fig. 10 shows the nonuniform mesh for half the domain. Equi-stress function lines and the shear stresses are shown for $f = 1$, 2 *and* 4 in Figs. 11–13. The propagation of plastic region with increasing f can be seen from Fig. 14.

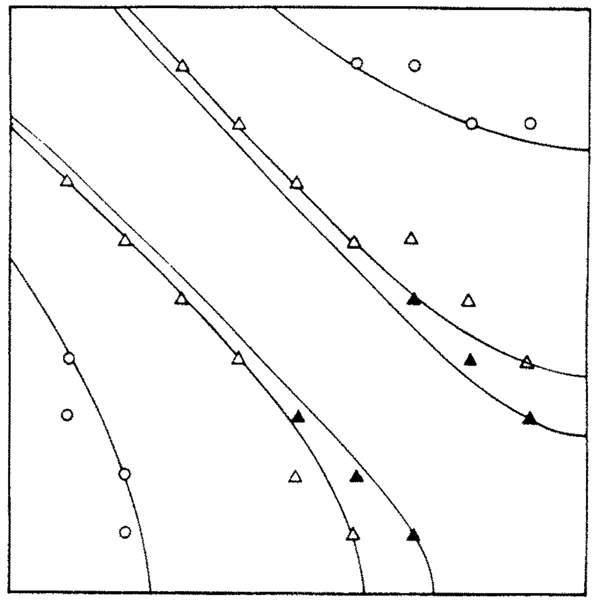


Fig. 6. Plastic regions for various values of f.

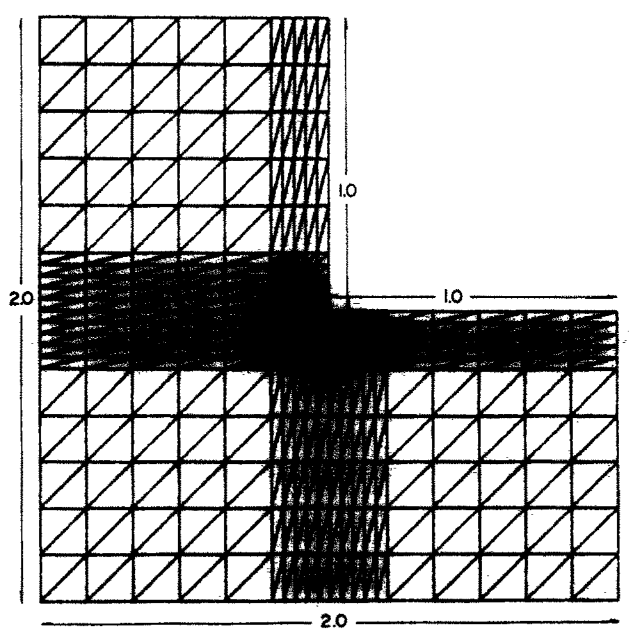


Fig. 7(a).

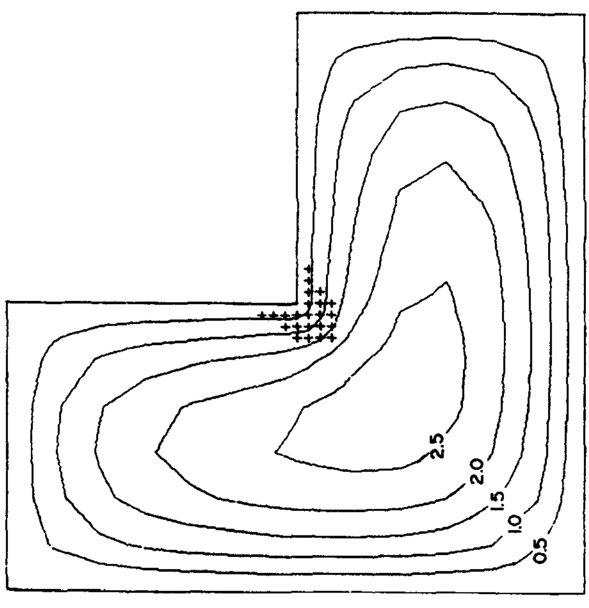


Fig. 7(b). $f = 2.0$.

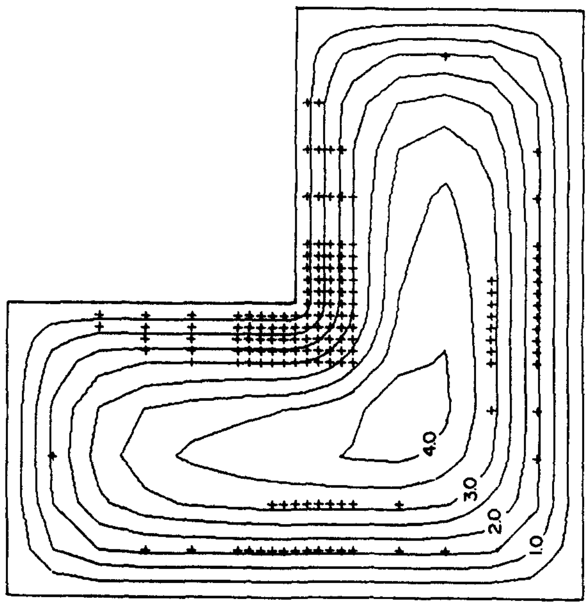


Fig. 7(c). $f = 4.0$.

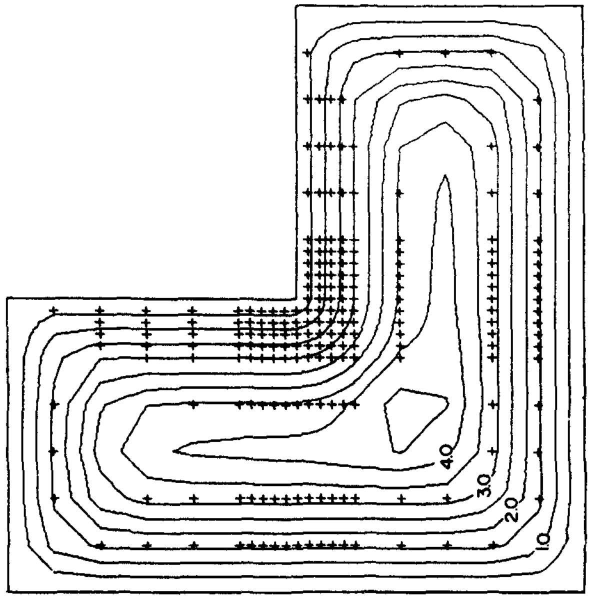


Fig. 7(d). $f = 6.0$.

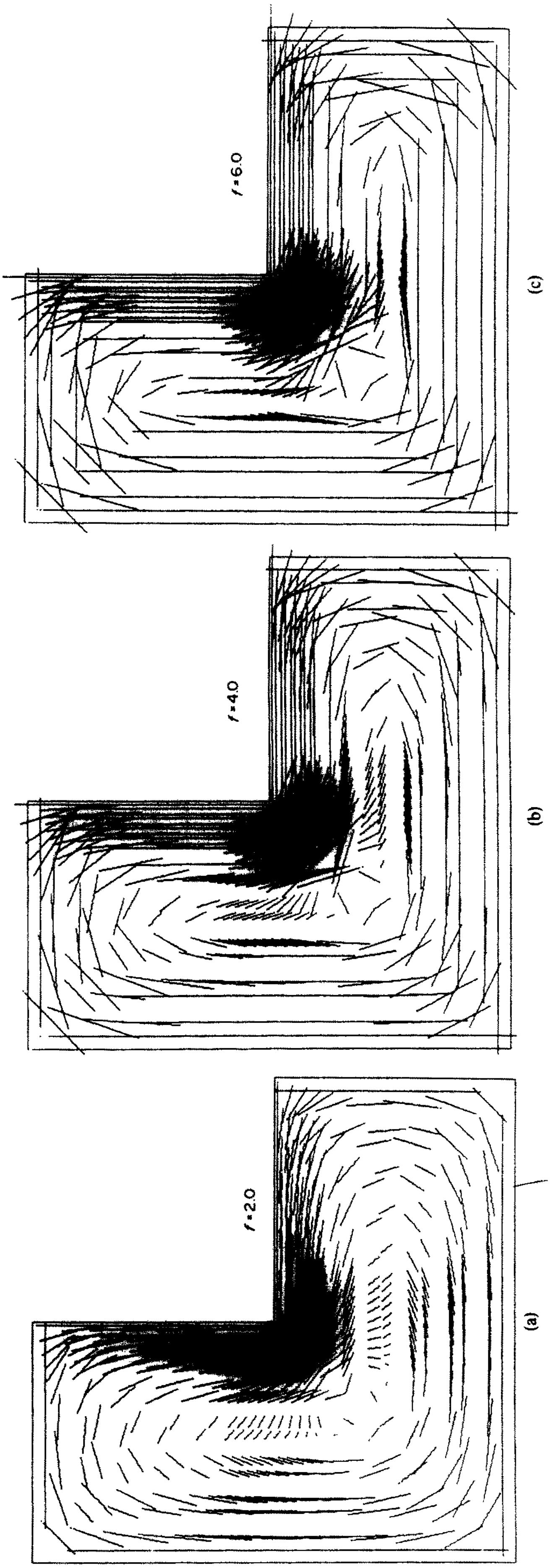


Fig. 8. The shear stress distribution.

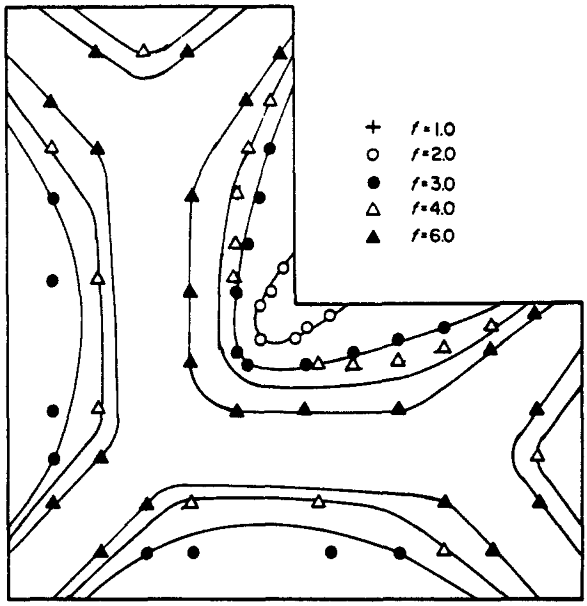


Fig. 9. Plastic regions for various values of *f*.

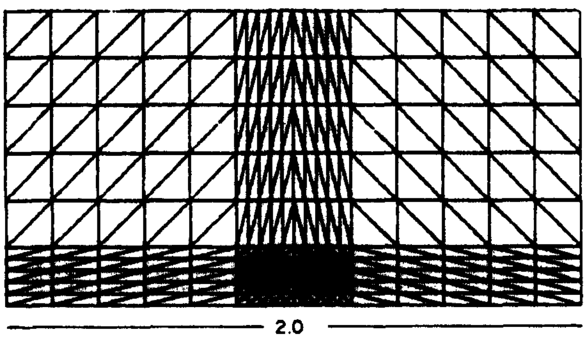


Fig. 10.

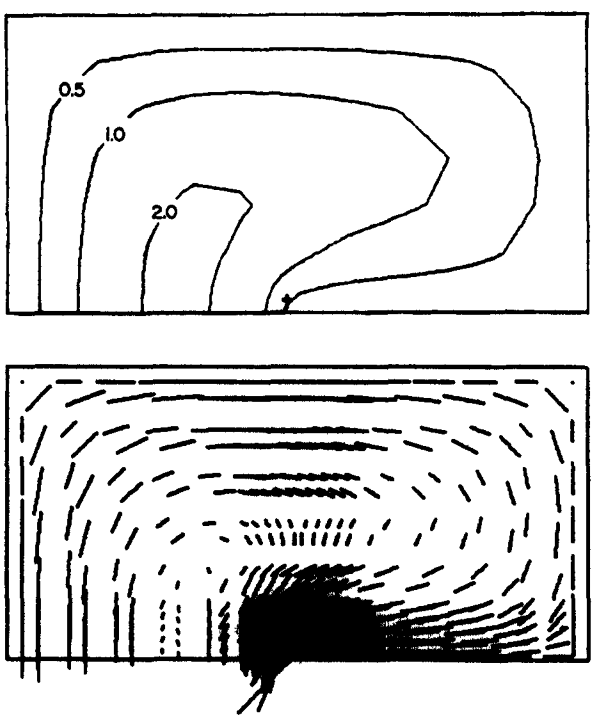


Fig. 11. $f = 1.0$.

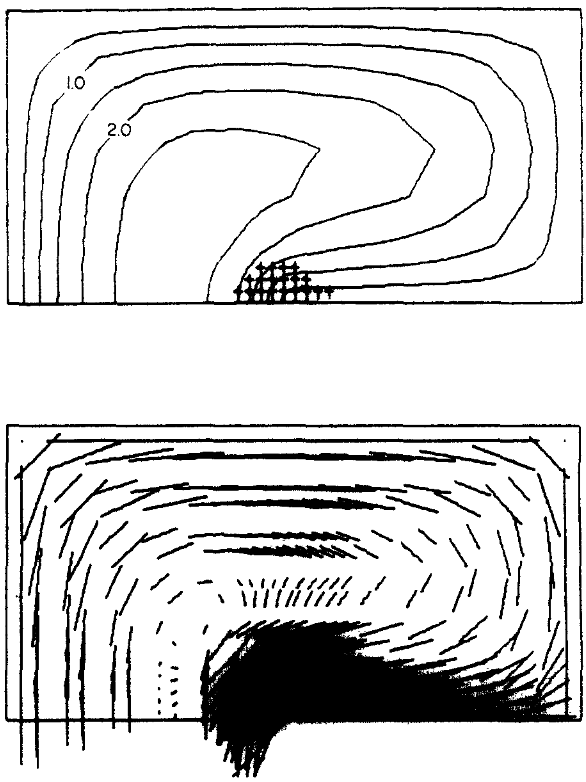


Fig. 12. $f = 2.0$.

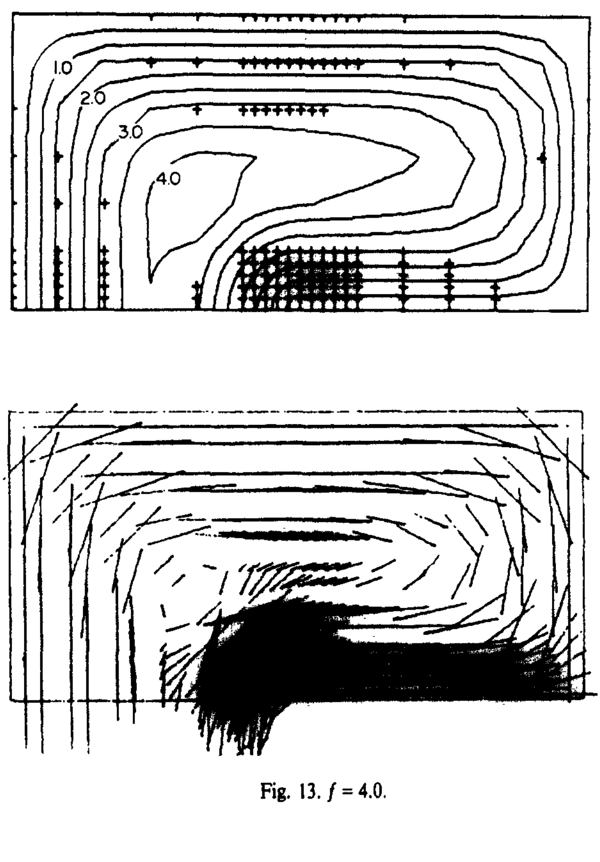


Fig. 13. $f = 4.0$.

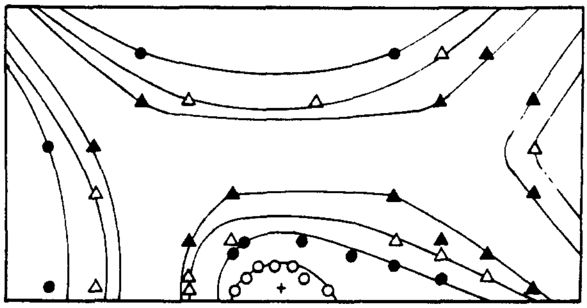


Fig. 14. Plastic regions for various values of f.

4. SUMMARY AND CONCLUSIONS

Here we presented various variational formulations of the elastic-plastic torsion problem. We chose for numerical calculation the projectional form of the formulation which is based on the proof of existence and uniqueness of solutions (see [12]). The fully-plastic eqn (28) is solved for ϕ_p in closed form by means of the theory of characteristics and then an iterative procedure is employed to solve the finite element analogue of eqn (9) for ϕ. A number of numerical examples are presented to show the feasibility and effectiveness of the present method.

As indicated in Section 3, the choice of the acceleration parameter ρ is very crucial in obtaining convergent solutions. A working value of ρ is obtained by trial in the present investigation. Lack of theoretical bounds, analogous to that established by Young[17] for positive definite linear equations, on ρ in the present case necessitated us to numerically investigate the influence of ρ on the convergence. Thus, there exists a need to find theoretical bounds on ρ for non-positive definite forms.

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APPENDIX

Numerical solution of $|\nabla\phi|^2 = \tau_p^2$ in $\Omega \subset R^2$

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - C^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = 0 \tag{A1}$$

where C is the wave speed. Let

$$u = u(t - \phi(\mathbf{x})) \tag{A2}$$

be the solution where $\phi(\mathbf{x})$ is a function to be determined. Let

$$\psi(\mathbf{x}, t) = t - \phi(\mathbf{x}).$$

Then, (A1) can be written as

$$\frac{\partial^2 u}{\partial \psi^2} - C^2 \sum_{i=1}^{2} \left\{\frac{\partial^2 u}{\partial \psi^2}\left(\frac{\partial \phi}{\partial x_i}\right)^2 - \frac{\partial u}{\partial \psi}\frac{\partial^2 \phi}{\partial x_i^2}\right\} = 0 \ (i = 1, 2).$$

Collecting the coefficients of $(\partial u/\partial \psi)$ and $(\partial^2 u/\partial \psi^2)$,

$$\left[1 - C^2 \sum_{i=1}^{2}\left(\frac{\partial \phi}{\partial x_i}\right)^2\right]\frac{\partial^2 u}{\partial \psi^2} + C^2\left(\sum_{i=1}^{2}\frac{\partial^2 \phi}{\partial x_i^2}\right)\frac{\partial u}{\partial \psi} = 0. \tag{A3}$$

It follows then that $\psi(\mathbf{x}, t) = \text{constant}$, are the characteristic lines, and that the coefficient of $(\partial^2 u/\partial \psi^2)$ must vanish:

$$1 - C^2 \sum_{i=1}^{2}\left(\frac{\partial \phi}{\partial x_i}\right)^2 = 0 \quad \text{or} \quad |\nabla \phi|^2 = \frac{1}{C^2}. \tag{A4}$$

Thus, $\phi(\mathbf{x}) = t - \psi(\mathbf{x}, t) = t + \text{constant}$ is the solution of (A4) at point $\mathbf{x}$ for a given t. Suppose that the wave starts at $t = 0$ from the boundary and propagates into the domain. The initial condition is $\phi(\mathbf{x}) = 0$ (which describes the equation for the boundary) and therefore the constant is zero. The function $\phi(\mathbf{x})$ denotes the time taken by the wave to travel from the boundary to a point $\mathbf{x}$. Since the wave speed is C, we must have

$$t = \phi(\mathbf{x}) = d(\mathbf{x}, \partial\Omega)/C, \quad C = 1/\tau_p \tag{A5}$$

where $d(\mathbf{x}, \partial\Omega)$ is the minimum distance from the boundary $\partial\Omega$ of the domain Ω to the point $\mathbf{x}$.

In the numerical scheme, the minimum distance from a given node (in the finite element mesh) to boundary (nodes) is computed, and then $\phi(\mathbf{x})$ is obtained only at discrete points, namely at the nodal points.